Extended First Harmonic Approximation in Case of LLCC Converters with Capacitive Output Filter

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Abstract—The analysis of resonant converters is in general more cumbersome than in case of PWM converters, as often a multitude of modes has to be regarded and resonant oscillations with durations impossible to calculate directly are encountered. Therefore, approximations are required in order to facilitate the design process. An extended approximation in the frequency domain for the steady-state solution of the multi-resonant LLCC converter with capacitive output filter is presented in this paper, with a new degree of simplification regarding the resulting closed-form solution. By means of this approximation, the design of the fourth-order resonant LLCC converter is significantly simplified, providing a tool for rapid simulation combined with a very high accuracy. The accuracy of the investigated approach was double-checked by means of exact calculations.

I. INTRODUCTION

A. Existing approaches

Concerning the analysis of resonant converters, a large number of publications has been published following different approaches in order to derive their steady-state solution [1-6]. In case of the fourth-order multi-resonant LLCC converter shown in Fig. 1, several resonant frequencies can be identified, with a high number of possible switching modes, making an exact analysis very tedious [7-9]. Furthermore, numerical problems are encountered as most often transcendental equations have to be solved in order to obtain the steady-state solution. Time-discrete models on the other hand can deliver the steady-state solution, with the disadvantage of high requirements regarding computing time and thus making parameter variations cumbersome.

In order to overcome the problems associated with solutions in the time domain, several authors have investigated approximations in the frequency domain, with the First Harmonic Approximation (FHA) from [10] as the most known approach. With regard to resonant converters of higher order, this approximation is suffering from inaccurate results, whereas the basic characteristics of resonant converters can be well understood by means of this approach [11-13]. The origin of these inaccuracies can be found in the equivalent model of the output rectifier bridge, with the regular FHA deriving a simple equivalent resistor representing the diode bridge together with the output filter and the load. In order to increase the accuracy of this approximation, extensions have been published, deriving equivalent impedances for the nonlinear load (rectifiers + output filter + load) together with the parallel reactances of the resonant tank [14-16]. One of the first approaches [17] dealing with this approach for the LLCC limited this approach to frequencies with the parallel capacitor dominating the impedance across the input of the rectifier bridge. A more general approach was published in [18], resulting in very good agreements with measurements but with high numerical requirements. Based on the latter approach a refined closed-form solution is derived in this paper, resulting in only one transcendental equation. These findings significantly reduce the numerical complexity of the corresponding implementation within a design program. For regular converter designs, the derived solution is valid for a very wide frequency range.

B. Assumptions

In order to analyze the converter, simplifications and assumptions are necessary. All semiconductors and passives are assumed to be ideal. The MOSFET bridge is driven with a duty-cycle of 50%, neglecting dead times. All passive components are assumed to be loss-free. The actual transformer is modeled by a cantilever model with its equivalent components $L_s$, $L_h$ and $n$. The voltage of the parallel resonant tank across $C_p$ is rectified by a full-wave rectifier, with the output capacitor assumed to be large enough for $V_o$ to be ripple-free.

![Figure 1. Schematic of the fourth-order multi-resonant LLCC converter with capacitive output filter, driven by a half-bridge MOSFET configuration.](image-url)
For the operation of the converter, only modes with two subintervals are considered, which in fact covers the most part of the reasonable operating points of the converter.

C. Normalization

In order to simplify the obtained results, a normalization is carried out. Due to this normalized representation, the derived results can be easily applied to any converter design. Following the nomenclature of previous authors, normalized quantities are represented by replacing their letter while keeping the same subscript. The letter $V$ denoting voltages is replaced by the letter $M$. $I$ denoting currents is replaced by $J$. Table I shows the resulting normalized terms. The normalization voltage can be chosen freely, but if $V_i$ is selected as normalization voltage, the dc conversion ratio $M = V_o/V_i$ is automatically included in the derived results.

II. THE EXTENDED FIRST HARMONIC APPROXIMATION

A. Basic idea

Following the ideas of the regular FHA, it can be stated that for a large amount of operating points the series resonant tank current $i_s(t)$ is quasi-sinusoidal. An actual measurement of input waveforms with a switching frequency above the resonant frequency determined by $L_s$ and $C_s$ in case of the LLCC is shown in Fig. 2 (a), with the major part of the energy at the input indeed being transferred by the fundamental frequency. In contrast to that, the calculated waveforms at the converter’s output are given in Fig. 2 (b), with a clipped voltage waveform typical for resonant converters with a parallel resonant capacitance across the diode bridge. As can clearly be seen, the output current $i_{pk}(t)$ delivering energy to the load is by far not sinusoidal for capacitive output filters as illustrated in Fig. 2 (b). However, this is an assumption the regular FHA depends on. Furthermore, a significant phase angle between the fundamental components of the involved parallel voltage $v_p(t)$ and the output current $i_{pk}(t)$ is observed, with the regular FHA completely neglecting this phase shift. Thus the proposed approach focuses on this phase shift in order to model the interaction between the resonant tank and the nonlinear output stage more accurately as demonstrated in [18]. Thus, instead of an equivalent resistance, an equivalent impedance $Z_{ac}$ is derived in order to model the nonlinear rectifier in combination with the load.

B. Analysed mode

With regard to the derivation of this equivalent impedance $Z_{ac}$, a time domain analysis has to be carried out in the first place. The simplified equivalent converter model for this analysis is given in Fig. 3. As discussed above, the eFHA relies on the same assumption of a sinusoidal current $i_{s}(t)$ in the series resonant tank with

$$i_s(t) = i_s \sin(\omega t - \nu)$$

as in case of the regular FHA, with the angle $\nu$ representing the phase shift of the input current $i_s(t)$ vs. the parallel voltage $v_p(t)$. Under this assumption, the switching action of the input bridge is directly modeled by the first harmonic of the input current. Thus only two switching states can occur within the eFHA which are determined by the output rectifiers. If energy is transferred to the output, two of the four rectifier diodes conduct. In between two switching states with conducting rectifiers, a resonant subinterval has to occur in order to recharge the parallel capacitance $C_p$ from $+V_o$ to $-V_o$ or the other way round. During this subinterval all four diodes are reverse-biased. The waveform of the parallel voltage $v_p(t)$ during this subinterval is given by

$$v_p(t) = A \sin(\omega_{p} t - \theta) + B \cos(\omega t - \nu) \quad 0 \leq t \leq t_i$$

with

$$B = n / C_p \cdot \omega_{p} / (\omega_{p}^2 - \omega^2)$$

The current $i_{pk}(t)$ flowing through the parallel capacitor is given by the derivation of (2) with

$$i_{pk}(t) = \dot{I}_s \sin(\omega t - \nu) + i_s(t)$$

Figure 2. LLCC waveforms; (a) measured input waveforms, (b) schematic output waveforms

![Figure 2](image_url)

Figure 3. Schematic of the fourth-order multi-resonant LLCC converter with capacitive output filter, driven by a half-bridge MOSFET configuration
\[ i_p(t) = C_p \left[ A \omega_p \cos(\omega_p t - \theta) - B \omega_p \sin(\omega_p t - \nu) \right] \quad 0 \leq t \leq t_i . \quad (4) \]

The magnetizing current \( i_p(t) \) can be expressed in terms of the source current \( i_s(t) \) and the primary transformer current with

\[ i_s(t) = i_p(t) - i_d(t) . \quad (5) \]

During subintervals with conducting diode bridge, the parallel voltage is clamped to the output voltage, resulting in

\[ v_p(t) = -V_o \quad t_i \leq t \leq T_s / 2 \quad \text{and} \]

\[ i_p(t) = 0 \quad t \leq t \leq T_s / 2 . \quad (7) \]

The magnetizing current \( i_d(t) \) during this subinterval is a linear function with

\[ i_d(t) = i_s(t_i) - nV_o \cdot (t - t_i) / L_p \quad t_i < t \leq T_s / 2 . \quad (8) \]

The analysis discussed in this paper is limited to one mode consisting of four subintervals. This mode is most commonly encountered for a large range of configurations. Other modes, especially for switching frequencies below \( f_{so} \) with a higher number of subintervals are neglected. For switching frequencies above the resonant frequency \( f_{so} \) with the desirable feature of ZVS this mode is most likely to occur for regular converter designs. Therefore it is sufficient to limit the analysis to the aforementioned two subintervals, since the following two subintervals are anti-symmetric to the first two. Nevertheless, the described methodology can be applied to other operating points consisting of more than four subintervals accordingly.

### C. Steady-State analysis

The complete eFHA waveforms concerning the analyzed operating points are shown in Fig. 4. In order to derive the equivalent impedance \( Z_o \), by which the series branch is loaded with, the amplitude \( I_0 \) of the current source driving the resonant tank is assumed to be known in the first step. First of all, the unknown parameters \( A \), \( \theta \) and \( \nu \) have to be expressed in terms of the given variables. In this context it has to be noted that the time reference \( t_0 \) of the analysis is chosen as the moment in which the resonant capacitor \( C_p \) starts to be discharged from \( v_p(0) = V_o \). Thus the first equation describing the closed-form solution can be identified based on (2) with

\[ v_p\left(0^+\right) = v_p\left(0^-\right) \Rightarrow A \sin \theta = B \cos \nu - V_o . \quad (9) \]

The second subinterval ends at \( t_i \) when \( v_p(t) \) reaches the negative value of the output voltage, starting a clamped subinterval in the opposite direction with

\[ v_p(t_i) = -V_o \Rightarrow A \sin(\omega_p t_i - \theta) + B \omega_p \sin(\omega_p t_i - \nu) = -V_o . \quad (10) \]

In order to simplify the resulting equations, the switching frequency \( f_s \) is normalized with respect to the resonant frequency \( f_{so} \) of the parallel tank with the primary reflected parallel capacitance \( C_p/r_0^2 \) as shown in Table I. As illustrated in Fig. 4, the duration of the resonant subinterval is depicted as \( \epsilon = \omega t_i \) and (10) can be rewritten as

\[ A \sin(\epsilon/F - \theta) + B \cos(\epsilon - \nu) = -V_o . \quad (11) \]

The second state variable describing the eFHA waveforms is the magnetizing current \( i_d(t) \) defined in (5), which has to be steady at \( t = t_i \), resulting in

\[ i_d(t) = \dot{I_s} \sin(\epsilon - \nu) - \dot{I_s} - C_p / n \cdot [A \omega_p \cos(\epsilon/F - \theta) - B \omega_p \sin(\epsilon - \nu)] . \quad (12) \]

Thus the switching condition of the network is given by the rectifier current \( i_d(t) \). As illustrated in Fig. 4, the output current delivering energy to the load vanishes at \( t = 0 \) or \( t = T_s / 2 \), respectively. Based on this condition a further equation can be derived with

\[ i_d(t) = 0 \Rightarrow A = -BF \sin \nu / \cos \theta . \quad (13) \]

Substitution of (13) in (9) yields

\[ \theta = \arctan\left[ \frac{V_o / (B \cos \nu) - 1}{F \tan \nu} \right] . \quad (14) \]

Taking (9), (13) and (14) into account, (11) can be simplified

\[ \frac{V_o}{B} = F \sin \nu \sin(\epsilon/F) \cos \theta + B \cos(\epsilon - \nu) = -V_o \]

\[ \Rightarrow F \frac{V_o}{B} = \frac{B}{F} \sin \nu \sin(\epsilon/F) \cos \theta + B \cos(\epsilon - \nu) + 1 \cos(\epsilon/F) + 1 . \quad (15) \]

Thus, for a given eFHA source current amplitude defining \( B \) in (3), the output voltage can be calculated by means of (15). The quotient \( V_o / B \) can now be substituted (14) and the angle \( \theta \) is obtained as a function of the normalized switching frequency, the duration of the resonant subinterval \( \epsilon \) as well as the phase angle \( \nu \) between \( v_p(t) \) and \( i_d(t) \) with

![Figure 4. eFHA waveforms](image-url)
\[ \theta = \arctan \left[ \frac{-(1 + \cos \epsilon) \tan \nu + F \sin (\epsilon / F) - \sin \epsilon}{F \cos (\epsilon / F) + 1} \right] \] (16)

In order to describe the closed-form solution of the investigated eFHA, the angles \( \epsilon \) and \( \nu \) remain as the last unknown variables, if the amplitude \( \hat{i} \) of the source current is assumed to be known. For the purpose of eliminating one of these unknown angles, a last equation is derived based on the switching condition at \( t = T/2 \) with

\[ i_b(T/2^+) = 0 \quad \Rightarrow \quad \hat{i}_s \sin (\pi - \nu) = i_{\text{sh}} - \frac{nV_s}{\omega_0 L_0} (\pi - \epsilon) \] (17)

Together with (3), (12), and (16) one obtains

\[ \sin \nu [1 - F^2(1 + \cos \epsilon / F)] \sin (\epsilon / F) + + V_s / B \left[\pi - \epsilon - F \sin (\epsilon / F)\right] + F \sin (\epsilon / F) \cos \nu = 0. \] (18)

Substitution of (15) in (18) yields

\[ \nu = \arctan \left(N/Z\right) \] (19)

with \( K = [\pi - \epsilon - F \sin (\epsilon / F)] / [\cos (\epsilon / F) + 1] \) and \( N = -F \sin (\epsilon / F) + \sin \epsilon - K \cos (\epsilon / F) - \cos \epsilon \) and

\[ Z = 1 - F^2 \left[1 + \cos \left(\frac{\epsilon}{F}\right)\right] + \cos \epsilon + K \left[F \sin \left(\frac{\epsilon}{F}\right) - \sin \epsilon\right]. \]

As shown in (19), the phase angle \( \nu \) is solely a function of the normalized switching frequency and the duration of the resonant subinterval \( \epsilon \). Thus, the steady-state of the converter can be described for a given combination of \( \epsilon \) and \( \hat{i}_s \). The derivation of these parameters for a given converter configuration and operating point is discussed in the next sections.

**D. Equivalent impedance \( Z_{\text{eq}} \)**

As discussed in the previous sections, the steady-state waveforms under eFHA assumptions are determined by the angle \( \epsilon \) and the amplitude \( \hat{i}_s \) of the driving current source \( i_b(t) \). Following the methodology described in [14-16,18], an equivalent impedance is derived in the next step, representing all parallel components \( (L_{\text{ph}}, C_{\text{ph}}/n^2) \) as well as the nonlinear rectifier bridge together with the load resistor. With a sinusoidal series current as the basic eFHA assumption, the active power delivered by the input is transferred by the first harmonics of \( nV_s(t) \) and \( i_b(t) \) only. Thus the impedance \( Z_{\text{eq}} \) can be expressed as

\[ Z_{\text{eq}} = n \overline{L_{\text{ph}}}/\overline{I_{\text{ph}}} \quad \text{with} \quad \overline{L_{\text{ph}}} = -j \overline{I_{\text{ph}}} e^{-j\nu}, \] (20)

with the index 1 denoting the phasor of the first harmonic. The phasor of the first harmonic of the series current \( i_b(t) \) is directly given by the definition of the corresponding waveform in (1). The phasor of the parallel voltage \( v_s(t) \) has yet to be determined. A Fourier series expansion is carried out, neglecting all higher harmonics of the parallel voltage similar to the regular FHA. Nevertheless, the phase shift between the two phasors determining \( Z_{\text{eq}} \) in (20) is taken into account.

Considering the anti-symmetric waveform of the converter’s state variables, one obtains

\[ \overline{V_s(t)} = \frac{2}{T_s} \left[ \int_0^{T_s/2} A \sin (\omega t - \theta) \cdot e^{-j\omega t} \, dt + \int_0^{T_s/2} B \cos (\omega t - \nu) \cdot e^{-j\omega t} \, dt \right] \] (21)

These three integrals can be solved in terms of the unknown duration of the resonant subinterval \( \epsilon \). The final result can be obtained in terms of

\[ Z_{\text{eq}} = f(\epsilon, \nu, F, R_0) \] (22)

with \( R_0 \) representing the characteristic impedance of the parallel tank which is given by

\[ R_0 = \sqrt{L_s / (C_p \cdot n^2)} = 2 \pi / (\nu_0 L_0) = (\omega_p C_p / n^2)^{1/2}. \] (23)

Keeping (19) in mind, the angle \( \nu \) can be expressed in terms of \( \epsilon \), thus it is found that the equivalent eFHA impedance \( Z_{\text{eq}} \), given in (22) is a function of \( R_0 \), the normalized switching frequency \( F \) as well as duration of the resonant subinterval represented by the angle \( \epsilon \) alone. The influence of the series tank is not directly visible, as it is inherently included in the derivation of \( \epsilon \). In order to account for the impact of the series tank on the input impedance of the converter, a frequency domain analysis follows as described in the next section.

**E. Frequency domain analysis**

Based on (22) and the discussed eFHA assumptions, the equivalent circuit shown in Fig. 5 is obtained. This equivalent circuits represents the complete converter in the frequency domain. The amplitude \( \hat{i}_s \) of the source current is excited by the first harmonic of the square-wave voltage \( v_s(t) \) generated by the half-bridge shown in Fig. 1. The input impedance limiting this current consists of the series tank as well as the derived equivalent impedance \( Z_{\text{eq}} \) representing the parallel tank together with the nonlinear load. In order to analyze a given operating point, this amplitude has to be calculated. The amplitude of the phasor of the first harmonic of \( v_s(t) \) is given by a Fourier series expansion of a square-wave signal with

\[ \overline{V_s} = 4 V_s / \pi. \] (24)

The amplitude \( V_s \) of the input square wave voltage is identical to the DC input voltage in case of a full-bridge configuration, in case of a half bridge switch configuration \( V_s = V/2 \) is obtained, with the DC component of \( v_s(t) \) as additional voltage drop across the series capacitor \( C_s \). The analysis of the
The equivalent circuit of Fig. 5 finally yields an expression for the current amplitude $I_0$ as a function of the duration of the resonant subinterval $\epsilon$ with

$$I_0 = \left[ J_0 \right] = \left| \frac{4V_g}{\pi} \right| j/\omega L_s + \left| 1/j\omega C_s + Z_m \right|$$  \hspace{1cm} (25)

Eq. (25) can be rewritten in normalized form as

$$J_s = \frac{4/\pi}{jF \lambda + \frac{1}{jF \zeta} + \frac{Z_m}{R_0}}$$ \hspace{1cm} (26)

Substituting (19) and (22) in (26), one obtains the missing link to the steady-state analysis of the eFHA within the time domain. As already mentioned, the converter’s output voltage and the input current are linked by (15). This equation is thus used in order to derive the last unknown variable $\epsilon$.

**F. Output plane**

The output plane of a resonant converter [6] is a graphical representation of the possible combination of given output voltage $M_o$ vs. the corresponding output current. Keeping in mind that $M_o$ is a given variable from this point of view, (17) can be rearranged as

$$J_o = \frac{nM_o \left| (1/F - F) \right| \left[ \cos(\epsilon/F) + 1 \right]}{F \sin \nu \sin(\epsilon/F) \cos \nu - \cos(\epsilon - \nu)}$$  \hspace{1cm} (27)

By substitution of (27) in (26), the steady-state solution describing the converter’s output is found. Since $M_o$ is known in terms of the output plane, a transcendental equation is found for the duration of the resonant subinterval $\epsilon$. A solution for this equation has to be found in the range of $0 \leq \epsilon \leq \pi$ by numerical means. Since the range of valid values for $\epsilon$ is limited, the solution of the obtained transcendental equation can be easily found.

In order to derive the output plane as diagram describing the converter’s output characteristic it is necessary to calculate the DC output current $I_o$ by integrating the output current $i(t)$. As there cannot be a DC current through the output filter capacitor $C_o$, one obtains

$$J_o = \frac{n}{\pi} \left[ J_0 \left| (\pi - \epsilon) \sin \nu - \cos \nu - \cos(\epsilon - \nu) \right] \right.$$

$$\left. + nM_o \left( \pi - \epsilon \right) \left| 2F \right| \right]$$  \hspace{1cm} (28)

With the determined value for the angle $\epsilon$, the amplitude of the source current is calculated by means of (26). With a known value for $M_o$, $J_o$ can then be calculated using (28).

**G. Control plane**

In terms of controlling the output voltage within resonant converters, the switching frequency of the input bridge is used in order to adapt the converter’s characteristic to input voltage or load variations. Thus the dc characteristics of resonant converters are often illustrated by means of the resulting output voltage $M_o$ for a given load resistance vs. switching frequency $F$. Keeping the chosen normalization in mind, one obtains

$$J_o = M_o \sqrt{Q} \text{ with } Q = R_o / R_0.$$  \hspace{1cm} (29)

Equation (29) together with (27) and (19) can be substituted in (28) resulting in a transcendental equation for the angle $\epsilon$ in terms of the control plane. For a given value of the load represented by $Q$, the corresponding output voltage for this load value can then be calculated by means of (27) in combination with (26) which has to be solved for $M_o$.

**III. VERIFICATION**

**A. Output plane**

As the investigated approach in this paper is based on ideal assumptions, ideal analysis methods were used in order to evaluate its accuracy. The exact solution with respect to the output plane for the ideal multiresonant LLCC converter shown in Fig. 1 was obtained by means of a time-domain analysis. For this purpose, several different modes have to be analyzed, with each mode having a set of transcendental equations that has to be solved in order to obtain the steady-state solution. It is evident that this approach is by far more time-consuming than the investigated eFHA. An exemplary comparison of output planes for $\zeta = \lambda = 1.5$ is shown in Fig. 6 with the corresponding curves for the angle $\epsilon$ in Fig. 9.

For all displayed frequencies, the obtained eFHA results are very close to the exact solution. In case of the predicted no load voltage with $J_o = 0$, the eFHA calculates a lower value. The corresponding value for the duration $\epsilon$ under no-load conditions is $\pi$, as no power is transferred to the output and the resonant subintervals continue for the complete switching cycle. For short-circuit conditions with $M_o = 0$ on the other hand, the predicted output current is in very good agreement with the results obtained by the exact time domain model. The duration of the resonant subintervals is small for points of operation with high output currents as the parallel capacitance is recharged quickly. Altogether it can be stated that the eFHA is a suitable tool in order to calculate the converter’s output characteristics.

![Figure 6. Comparison of eFHA vs. exact output plane with $\zeta = \lambda = 1.5$](image)
B. Control plane

The verification of the results in terms of the control plane was done by means of SPICE. The voltage conversion ratio as a function of the normalized switching frequency is shown in Fig. 7. Additionally, FHA results are displayed in order to compare these results to the obtained eFHA predictions. Depending on the load of the converter represented by the quality factor $Q$, notable differences occur between the regular FHA results and the exact results obtained by SPICE. In contrast to that, the obtained eFHA results are visibly closer to the exact solution. The same behaviour is observed in terms of other important design quantities such as the rms series inductor current shown in Fig. 8.

CONCLUSIONS

An advanced method for the rapid calculation of multi-resonant converters is described in this paper. Offering a significantly reduced mathematical complexity compared to other approaches, this methodology is very well suited to be implemented within calculation routines for the design of resonant LLCC converters. Furthermore, numerical problems are minimized since the resulting closed-form solution can be simplified down to one transcendental equation. The accuracy of the results obtained by the investigated approach was double-checked in comparison to exact modeling approaches and a very good agreement was observed.

REFERENCES